Int. J. Solids Structures, 1974, 10, pp. 149–160. Pergamon Press. Printed in Gt. Britain.

ON THE STRUCTURE OF THE THEORY OF VISCOPLASTICITY†

DAVID W. NICHOLSON‡ and ARIS PHILLIPS§

Department of Engineering and Applied Science, Yale University, Connecticut 06520

(*Received* 24 *April* 1972; *revised* 29 *May* 1973)

Abstract—In various formulations of plasticity, there is evident a structure embracing several features, including inviscidity, a yield condition, and a constitutive inequality. By means of these features the constitutive equations of plasticity are derived. In the present paper we introduce a viscoplastic counterpart of the constitutive inequality of plasticity, and we consider its physical significance. We also present a theory of viscoplasticity having a structure similar to that of plasticity and its relation with the Hohenemser–Prager prototype of viscoplastic constitutive relations is considered.

1. INTRODUCTION

It is widely held that many materials when subjected to dynamically applied loads exhibit rate effects during yielding. Due to inviscidity, the theory of plasticity is unsuited for analysis of such behavior. One approach taken to achieve a satisfactory formulation has been to generalize plasticity to cases of rate influence. One such generalization has been provided in various forms of the theory of viscoplasticity.

The foundations of the theory of viscoplasticity can be considered to have been set by Bingham[1], Hencky[2] and Hohenemser and Prager[3]. In these early versions viscoplastic deformation occurs when the magnitude of the stress vector exceeds some critical value which is a material constant. Furthermore, the rate at which viscoplastic deformation occurs depends in a linear manner on how much the critical value is exceeded. Perzyna[4] and Phillips and Wu[5] achieved generalizations of the previous formulations by proposing more general conditions for viscoplastic deformation and a more general relationship of the rate of viscoplastic deformation to the amount by which the condition for viscoplastic deformation is exceeded.

It should be clearly understood that viscoplasticity is not the most general theory which can be formulated for dynamic plastic phenomena. In the above named formulations, for example, the rate of viscoplastic deformation is independent of the stress rates. More general assumptions have been introduced by Lubliner[6] and Cristescu[7], in which the inelastic deformation consists of an inviscid part and of a stress–rate independent part.

In various formulations of plasticity (see for example Prager[8], Drucker[9] and Phillips and Eisenberg[10]) there is evident a structure embracing several features including inviscidity, a yield condition, and a constitutive inequality. By means of these features the

† This research has been supported by the National Science Foundation under a grant with Yale University.

‡ Currently Senior Physicist, Research Division, Goodyear Tire and Rubber Company, Akron, Ohio.

§ Professor of Engineering and Applied Science.

constitutive equations suitable to describe plastic behavior are derived. These features also imply that in plasticity the inelastic strain rate is normal to a particular surface in stress space.

By contrast, the existing versions of viscoplasticity do not contain a constitutive inequality. Instead, a normality condition is assumed, presumably on the grounds of simplicity.

There are two main objectives in the present paper. The first is to introduce a viscoplastic counterpart of the constitutive inequality of plasticity rather than postulate a normality condition. The second is to present a theory of viscoplasticity having a structure similar to that of plasticity.

2. CONSTITUTIVE INEQUALITY IN PLASTICITY

In the following our attention will be restricted to infinitesimal strains and temperature independent deformations. All functions to be defined in this paper will be assumed continuously differentiable to as high an order as is necessary. The strain is given in terms of displacements by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{1}$$

We introduce elastic and inelastic parts of the strain through the kinematic decomposition

$$\varepsilon_{ij}{}^{i} = \varepsilon_{ij} - \varepsilon_{ij}{}^{el}, \tag{2}$$

from which we obtain

$$\dot{\varepsilon}_{ij}{}^{i} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}{}^{el}, \tag{2a}$$

where

$$\dot{\varepsilon}_{ij}{}^{el} = \frac{\dot{\sigma}_{ij} - \delta_{ij}\dot{\sigma}_{kk}/3}{2\mu} + \frac{\delta_{ij}\dot{\sigma}_{kk}}{9K}, \tag{3}$$

and μ and K are the elastic shear and the elastic bulk modulus, respectively. As expressed in equation (3), the elastic strain rate is governed by Hooke's law.

In plasticity the state of a material element undergoing plastic deformation is characterized by the quantities

$$[\sigma_{ij}, \varepsilon_{ij}{}^{i}, k] \leftrightarrow \text{state } S, \tag{3a}$$

where k is a scalar quantity to be selected to represent dependence on the history of inelastic strain. It is inherent in the meaning of the history parameter k that

$$\dot{k} = 0 \quad \text{whenever} \quad \dot{\varepsilon}_{ij}{}^{i} = 0.$$

If necessary, more than one history parameter may be introduced. The independent state variable is considered to be the stress, so that inelastic strain and the history parameter are viewed as dependent state variables. For simplicity, we will regard two states as identical if their corresponding state variables are equal.

The constitutive equations are assumed in the form

$$\dot{\varepsilon}_{ij}{}^{i} = g_{ij}(S, \dot{\sigma}_{pq}), \tag{4}$$

and

$$\dot{k} = h(S, \dot{\varepsilon}_{lr}{}^{i}). \tag{5}$$

Substituting equation (4) into equation (5) we obtain

$$\dot{k} = l(S, \dot{\sigma}_{pq}). \tag{6}$$

Equations (4) and (6) express the rates of the dependent state variables as functions of the state and the rates of the independent state variables. With regard to equation (4), a restriction on the stress rate dependence is introduced by assuming plastic deformation to be inviscid. Then equation (4) is homogeneous of order one in the time rates; i.e.

$$g_{ij}(S, \psi\dot{\sigma}_{pq}) = \psi g_{ij}(S, \dot{\sigma}_{pq}),$$

where ψ is arbitrary.
A sufficient condition for inviscidity is linearity:

$$g_{ij}(S, \dot{\sigma}_{pq}) = a_{ijpq}(S)\dot{\sigma}_{pq}. \tag{7}$$

Therefore

$$\dot{\varepsilon}_{ij}{}^{i} = a_{ijpq}(S)\dot{\sigma}_{pq}. \tag{8}$$

We now state the yield conditions for plastic deformation. Suppose that at some time t_0 the inelastic strain is $\varepsilon_{ij}{}^{i}$ and the history parameter is k but σ_{ij} need not be the actual stress. Consider a function $G(\sigma_{ij}, \varepsilon_{ij}{}^{i}, k)$ which, for a particular set of values $\sigma_{ij}{}^{*}$ comprising a closed surface in stress space, takes the zero value:

$$G(\sigma_{ij}{}^{*}, \varepsilon_{ij}{}^{i}, k) = 0. \tag{9}$$

We postulate that such a function exists and that the surface represented by equation (9), the "plastic yield surface," encloses or contains all stress points that are compatible with $\varepsilon_{ij}{}^{i}$ and k, the actual values of the dependent state variables. The stresses $\sigma_{ij}{}^{*}$ are called the yield points. By convention we have $G(\sigma_{ij}, \varepsilon_{ij}{}^{i}, k) < 0$ if σ_{ij} is interior to the surface represented by equation (9).

The condition for a change in inelastic strain is that the actual stress point σ_{ij} be located on the plastic yield surface and be moving toward its exterior; i.e.

$$\dot{\varepsilon}_{ij}{}^{i}\dot{\varepsilon}_{ij}{}^{i} \neq 0 \quad \text{if} \quad G(S) = 0 \quad \text{and} \quad \frac{\partial G(S)}{\partial \sigma_{pq}}\dot{\sigma}_{pq} > 0. \tag{10}$$

(The case of perfectly plastic materials is not being considered.)

Once the yield condition and inviscidity are assumed, a wide range of choice remains concerning the constitutive equation for the inelastic strain rate in equation (4). This choice is commonly restricted by means of the introduction of an inequality (see for example Prager[8], Drucker[9] and Phillips and Eisenberg[10]). The inequality assures that an inelastic strain increment will not be directionally opposite the stress increment producing it, and is stated as

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}{}^{i} \geq 0. \tag{11}$$

Using equations (7) and (10), according to Prager[8], the following relation can be written for a_{ijpq}:

$$a_{ijpq} = a_{ij}\frac{\partial G}{\partial \sigma_{pq}}. \tag{12}$$

Imagine now two elements which are identical and in the same state at time t_0:

$${}_{(1)}S = {}_{(2)}S$$

Then

$$_{(2)}\left[a_{ij}\frac{\partial G}{\partial \sigma_{pq}}\right] = {}_{(1)}\left[a_{ij}\frac{\partial G}{\partial \sigma_{pq}}\right].$$

For times $t \geq t_0$ the two elements are subject to stress rates $_{(1)}\dot{\sigma}_{ij}$ and $_{(2)}\dot{\sigma}_{ij}$ which are arbitrary. Then, using equation (12) it can be proved that inequalities (11) and (13) are equivalent:

$$[_{(2)}\dot{\sigma}_{ij} - {}_{(1)}\dot{\sigma}_{ij}][_{(2)}\dot{\varepsilon}_{ij}{}^{i} - {}_{(1)}\dot{\varepsilon}_{ij}{}^{i}] \geq 0. \tag{13}$$

(Derivatives are evaluated in the limit at $(t - t_0) \to 0$, for $t \geq t_0$). We introduce now the symbol δ defined through the following relation

$$\delta x = x(t) - x(t_0).$$

Then for a sufficiently small time interval δt we have

$$\delta x \cong \dot{x}^* \delta t,$$

where

$$\dot{x}^* = \lim_{\delta t \to 0} [\dot{x}(t)]$$

and in that case inequality (13) can be replaced by inequality (14)

$$[_{(2)}\delta\sigma_{ij} - {}_{(1)}\delta\sigma_{ij}][_{(2)}\delta\varepsilon_{ij}{}^{i} - {}_{(1)}\delta\varepsilon_{ij}{}^{(1)}] \geq 0. \tag{14}$$

The equivalence of this last expression to inequality (13) may be seen by expanding the stress and strain increments in a Taylor's series and selecting the time increment δt sufficiently small that lowest order terms dominate. A similar derivation is presented in Naghdi[11]. Summarizing the previous discussion we observe that inequalities (11) and (13) and for sufficiently small δt inequality (14) are equivalent as the constitutive inequality for plastic deformation.

Inequality (14) warrants further discussion. Consider two identical bars which have been pulled in uniaxial tension such that they are in the same state at time t_0. Imagine that for times $t \geq t_0$ the two bars are subject to different arbitrary (one-dimensional) stress rates. Inequality (14) implies that for sufficiently short time intervals the bar with the higher stress must also have the higher strain.

An inequality similar to inequality (14) has been proposed by Drucker[12] as a generalization of stability criteria presented for plasticity (Drucker[9]). Its implications for uniqueness were also discussed. Here, given linearity (inviscidity) and the yield conditions of plasticity, inequality (14) arises in the first place as the formal equivalent to the constitutive inequality of plasticity.

On the grounds that it is an inherently reasonable classification of material behavior it will later be assumed for viscoplasticity. In conjunction with the viscoplastic counterparts of linearity and of the yield conditions in plasticity, it will serve to derive a constitutive inequality for viscoplasticity.

An additional point of interest is that in plasticity the constitutive inequality may equivalently be stated in terms of one element (equation 11) or of two elements (equation 11 or equation 13). Later we will find that such an equivalence does not hold in viscoplasticity in the case of hardening.

The constitutive equations for plastic deformation may be derived by means of the foregoing relations. It should be noted that the normality of the inelastic strain rate to the plastic yield surface is obtained in the course of such derivations.

3. CONSTITUTIVE INEQUALITY IN VISCOPLASTICITY

In viscoplasticity, the concept of inelastic strain as introduced in equation (2) and the state characterization expressed in equation (3a) are retained. In the following we will: (a) define the dependence of the inelastic strain rate on the stress rate in viscoplasticity; (b) introduce yield conditions for viscoplastic deformation; and (c) present a viscoplastic counterpart of the constitutive inequality of plasticity.

In Hohenemser and Prager[3], Perzyna[4] and Phillips and Wu[5], versions of viscoplasticity have been presented in which the inelastic strain rate is taken to be independent of the time rates of the stress. We retain this feature and thereby assume that the equation for the inelastic strain rate (equation 4) is homogeneous of order zero in the stress rate.

Consequently, we have

$$g_{ij}(S, \psi\dot{\sigma}_{pq}) = g_{ij}(S, \dot{\sigma}_{pq}),$$

from which we conclude that g_{ij} is independent of $\dot{\sigma}_{pq}$ so that

$$\dot{\varepsilon}_{ij}{}^{i} = g_{ij}(S). \tag{15}$$

Similarly, we obtain

$$\dot{k} = l(S). \tag{16}$$

As previously stated, it is important to understand that viscoplasticity is not the most general theory it is possible to formulate for dynamic plastic phenomena. The zero order homogeneity is a restrictive assumption. It has been replaced in the formulations by Lubliner[6] and Cristescu[7] in which the inelastic strain rate is considered to decompose into an inviscid part and a stress rate independent part. It has also been argued that (Bell[13]) in many cases dynamic plastic phenomena may be analyzed by means of a purely inviscid theory.

We now state the conditions under which viscoplastic deformation occurs. Suppose that at some time t_0 the inelastic strain is $\varepsilon_{ij}{}^{i}$ and the history parameter is k but σ_{ij} need not be the actual stress. Consider a function $F[\sigma_{ij}, \varepsilon_{ij}{}^{i}, k]$ which takes the zero value for a particular set of points, say $\sigma_{ij}{}^{*}$, comprising a closed surface in stress space: equation

$$F[\sigma_{ij}{}^{*}, \varepsilon_{ij}{}^{i}, k] = 0. \tag{17}$$

For the actual stress F may in general be different from zero. By convention we have $F > 0$ if the stress point is exterior to the surface and $F < 0$ if it is interior to the surface.

As the condition for viscoplastic deformation we postulate that such a function exists and it assumes a positive value for the actual stress during viscoplastic deformation. Consequently, during viscoplastic deformation the actual stress is exterior to the surface represented by equation (17), which will be called "the reference viscoplastic yield surface." Let us call λ the positive value of F whenever we intersect into the function F the value of the actual stress point.

Equation

$$F(\sigma_{ij}, \varepsilon_{ij}{}^{i}, k) = \lambda\{\equiv F(S)\} \tag{18}$$

represents a closed surface enclosing the reference viscoplastic yield surface and it is called the 'dynamic viscoplastic loading surface'.

A considerable range of choice remains for the functions g_{ij} and F. In the existing versions of viscoplasticity this choice is restricted by assuming that the inelastic strain rate is normal to one of the viscoplastic surfaces. Instead of a normality postulate we wish to present an inequality which is a viscoplastic counterpart of the constitutive inequality of plasticity. As will be seen, this will allow assessing the meaning and restrictiveness of normality postulates. It will also permit presenting a theory of viscoplasticity with a structure similar to that evident in some versions of plasticity.

We consider various possible choices for the constitutive inequality of viscoplasticity. We first consider inequality (11). It is clearly unsuitable since it would, for example, exclude the case in which stress is decreasing while the strain is increasing but the strain rate is decreasing.

As a specific example consider the bilinear constitutive equation proposed by Wood and Phillips[14]

$$\dot{\varepsilon}^i = \eta\langle\sigma - \sigma_0 - k(\varepsilon^i - \sigma_0/E_0)\rangle \tag{19}$$

where

$$\langle\phi\rangle = \begin{cases} 0 & \text{if} \quad \phi \leqq 0 \\ \phi & \text{if} \quad \phi > 0. \end{cases}$$

It is clear that equation (19) would require

$$\ddot{\varepsilon}^i = \eta\langle\dot{\sigma}\left(1 + \frac{k}{E_0}\right) - k\dot{\varepsilon}^i\rangle$$

and for $\dot{\sigma} < 0$ and $\dot{\varepsilon}^i > 0$ would give $\ddot{\varepsilon}^i = 0$; that is $\ddot{\varepsilon}^i < 0$ would be impossible. This is obviously unsatisfactory.

As a second choice we consider inequality (14). Suppose we have two elements (1) and (2) which are identical and in the same state at time t_0:

$${}_{(1)}S = {}_{(2)}S$$

and therefore

$${}_{(1)}\varepsilon_{ij}{}^i = {}_{(2)}\varepsilon_{ij}{}^i$$

and

$${}_{(1)}k = {}_{(2)}k$$

and

$${}_{(1)}\sigma_{ij} = {}_{(2)}\sigma_{ij}.$$

We also have

$${}_{(1)}\dot{\varepsilon}_{ij}{}^i = {}_{(2)}\dot{\varepsilon}_{ij}{}^i$$

since in viscoplasticity the inelastic strain rate is a function of the state and for the two elements the states are equal at t_0.

For times $t > t_0$ the two elements are subject to stress rates ${}_{(1)}\dot{\sigma}_{ij}$ and ${}_{(2)}\dot{\sigma}_{ij}$ which are arbitrary. From elementary Taylor expansion we obtain

$$ {}_{(2)}\delta\varepsilon_{ij}{}^{i} - {}_{(1)}\delta\varepsilon_{ij}{}^{i} = [{}_{(2)}\dot{\varepsilon}_{ij}{}^{i} - {}_{(1)}\dot{\varepsilon}_{ij}{}^{i}]\delta t + [{}_{(2)}\ddot{\varepsilon}_{ij}{}^{i} - {}_{(1)}\ddot{\varepsilon}_{ij}{}^{i}]\frac{\delta t^2}{2} + \cdots $$

and a similar expansion for the stress increments. Restricting attention to sufficiently small time increments so that the lowest terms in the expansion dominate, we obtain from inequality (14)

$$ [{}_{(2)}\dot{\sigma}_{ij} - {}_{(1)}\dot{\sigma}_{ij}][{}_{(2)}\ddot{\varepsilon}_{ij}{}^{i} - {}_{(1)}\ddot{\varepsilon}_{ij}{}^{i}] \geq 0. \tag{20} $$

Applying inequality (20) to equation (19) we conclude that

$$ \eta[{}_{(2)}\dot{\sigma} - {}_{(1)}\dot{\sigma}]\left[({}_{(2)}\dot{\sigma} - {}_{(1)}\dot{\sigma})\left(1 + \frac{k}{E_0}\right) - k({}_{(2)}\dot{\varepsilon}^{i} - {}_{(1)}\dot{\varepsilon}^{i})\right] \geq 0 $$

which, since the inelastic strain rates at $t = t_0$ are equal, gives

$$ \eta \geqq 0. \tag{21} $$

Therefore inequality (20) implies that the reciprocal of the viscosity coefficient in equation (19) is positive. Also if inequality (21) is valid then inequality (20) will also be valid for the material given by equation (19).

The meaning of inequality (14) was discussed in a more general fashion at the end of the previous section. Here we assume it as a reasonable classification of material behavior. In plasticity, given inviscidity and the plastic yield conditions, it was shown to be equivalent to the constitutive inequality. For one dimensional deformations of a bilinear material in plasticity it is equivalent to the positiveness of the plastic modulus.

Given zero order homogeneity (equation 15) and the yield conditions of viscoplasticity (equations 17 and 18) inequality (20) and inequality (14) are equivalent. Inequality (20) will be called the constitutive inequality of viscoplasticity.

The constitutive inequality of viscoplasticity (20) does not hold in the Cristescu[7] and Lubliner[6] theories in which both an inviscid and a stress rate independent part of the inelastic strain rate are present. In such theories inequalities (14) and (20) are not equivalent. However, in such theories inequality (14) may be postulated.

Martin[15] has proposed for rigid non-hardening materials the inequality

$$ \dot{\sigma}_{ij}\ddot{\varepsilon}_{ij}{}^{i} \geqq 0 \tag{22} $$

which is a special case of inequality (20). It can easily be seen that this form is not appropriate in viscoplasticity for the case of hardening. Using equation (19) to exemplify this point, we find that

$$ \dot{\sigma}\ddot{\varepsilon}^{i} = \eta\dot{\sigma}^2 - k[\eta\langle \sigma - \sigma_0 - k(\varepsilon - \sigma_0/E_0)\rangle]\dot{\sigma} $$

which should be positive for arbitrary stress rates. However, the choice

$$ 0 < \dot{\sigma} < k[\sigma - \sigma_0 - k(\varepsilon - \sigma_0/E_0)] $$

does not satisfy this requirement.

More generally, from differentiation in equation (15) we obtain

$$ \dot{\sigma}_{ij}\ddot{\varepsilon}_{ij}{}^{i} = \frac{\partial g_{ij}}{\partial \sigma_{pq}}\dot{\sigma}_{ij}\dot{\sigma}_{pq} + \frac{\partial g_{ij}}{\partial \varepsilon_{pq}{}^{i}} g_{pq}\dot{\sigma}_{ij} + \frac{\partial g_{ij}}{\partial k} l\dot{\sigma}_{ij}. \tag{23} $$

Due to the arbitrariness of the stress rates and their independence from the present state we can always choose them leading to the violation of inequality (22) so long as the last two terms in equation (23) do not vanish.

So, in plasticity, and in viscoplasticity without hardening, the constitutive inequality may equivalently be expressed in terms of one material element or in terms of two material elements. But in viscoplasticity with hardening one element is not sufficient.

4. A BROAD CLASS OF ACCEPTABLE CONSTITUTIVE EQUATIONS

We now wish to identify acceptable constitutive equations for viscoplasticity.

We recall that

$$\dot{\varepsilon}_{ij}{}^{i} = g_{ij}(S) = g_{ij}(\sigma_{pq}, \varepsilon_{pq}{}^{i}, k).$$

Therefore

$$\ddot{\varepsilon}_{ij}{}^{i} = \frac{\partial g_{ij}}{\partial \sigma_{pq}} \dot{\sigma}_{pq} + \frac{\partial g_{ij}}{\partial \varepsilon_{pq}{}^{i}} \dot{\varepsilon}_{pq}{}^{i} + \frac{\partial g_{ij}}{\partial k} \dot{k}. \tag{24}$$

Substituting equation (24) into (20) we obtain

$$\frac{\partial g_{ij}}{\partial \sigma_{pq}} [{}_{(2)}\dot{\sigma}_{ij} - {}_{(1)}\dot{\sigma}_{ij}][{}_{(2)}\dot{\sigma}_{pq} - {}_{(1)}\dot{\sigma}_{pq}] \geq 0. \tag{25}$$

This result should be noted. The constitutive inequality of viscoplasticity by means of equation (25) has introduced a restriction on the possible choices for g_{ij}.

A sufficient condition for the constitutive inequality is achieved by setting

$$\frac{\partial g_{ij}}{\partial \sigma_{pq}} = {}_{(0)}L \, \frac{[\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}]}{2} + \sum_{r=1}^{N} {}_{(r)}L_{(r)}b_{ij(r)}b_{pq} \tag{26}$$

where

(a) ${}_{(0)}L(S) \geq 0$
(b) ${}_{(r)}L(S) \geq 0$, for all r
(c) ${}_{(r)}b_{ij}$ is a symmetric tensor valued state function of state.

Indeed, by introducing (26) into (25) we obtain after some algebra

$$\frac{\partial g_{ij}}{\partial \sigma_{pq}} \{{}_{(2)}\dot{\sigma}_{pq} - {}_{(1)}\dot{\sigma}_{pq}\}\{{}_{(2)}\dot{\sigma}_{ij} - {}_{(1)}\dot{\sigma}_{ij}\}$$
$$= {}_{(0)}L[{}_{(2)}\dot{\sigma}_{pq} - {}_{(1)}\dot{\sigma}_{pq}]^2 + \sum_{r=1}^{N} {}_{(r)}L[{}_{(r)}b_{ij}({}_{(2)}\dot{\sigma}_{ij} - {}_{(1)}\dot{\sigma}_{ij})]^2 \geq 0.$$

5. EXAMPLES

A special case satisfying equation (26) under certain conditions is

$$g_{ij}(S) = \eta \langle \phi(F) \rangle \Psi_{ij} \tag{27}$$

where

$$\langle \phi(F) \rangle = \begin{cases} 0, & F < 0 \\ \phi(F), & F \geq 0 \end{cases}$$

and where $\Psi_{ij} = \Psi_{ji}$. Indeed, differentiation in (27) yields (for $F(S) \geq 0$),

$$\frac{\partial g_{ij}}{\partial \sigma_{pq}} = \eta\phi(F)\frac{\partial \Psi_{ij}}{\partial \sigma_{pq}} + \eta\frac{\partial \phi}{\partial F}\frac{\partial F}{\partial \sigma_{pq}}\Psi_{ij}. \tag{28}$$

By writing

$$\eta\frac{\partial \phi}{\partial F}\frac{\partial F}{\partial \sigma_{pq}}\Psi_{ij} = {}_{(1)}L_{A\,(1)}b_{ij\,(1)}b_{pq} \tag{29}$$

and

$$\eta\phi\frac{\partial \Psi_{ij}}{\partial \sigma_{pq}} = {}_{(1)}L_{B\,(1)}b_{ij\,(1)}b_{pq} + \left[\frac{\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}}{2}\right]{}_{(0)}L + \sum_{r=1}^{N}{}_{(r)}L_{(r)}b_{ij\,(r)}b_{pq} \tag{30}$$

we see that these equations will be satisfied when

$$\Psi_{ij} = \partial F/\partial \sigma_{ij},\ {}_{(1)}b_{ij} = \partial F/\partial \sigma_{ij},\ {}_{(1)}L_A = \eta\frac{\partial \phi}{\partial F} \tag{31}$$

and

$$\eta\phi\frac{\partial^2 F}{\partial \sigma_{ij}\,\partial \sigma_{pq}} = {}_{(0)}L\left[\frac{\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}}{2}\right] + {}_{(1)}L_B\frac{\partial F}{\partial \sigma_{ij}}\frac{\partial F}{\partial \sigma_{pq}} + \sum_{r=2}^{N}{}_{(r)}L_{(r)}b_{ij\,(r)}b_{pq} \tag{32}$$

where

$${}_{(1)}L_A + {}_{(1)}L_B \geq 0.$$

The first equation (31) in conjunction with equation (27) is the normality condition. It should be remarked that equation (32) gives then a necessary condition on the yield surface F for normality to satisfy equation (26). Expression (27) together with $\Psi_{ij} = \partial F/\partial \sigma_{ij}$ was directly postulated by Perzyna[4] and was derived by Phillips and Wu[5] from another normality postulate. In neither case was there a discussion of the possible appropriateness of restrictions on the yield function such as expressed in equation (32).

In plasticity, given linearity and the yield conditions normality of the inelastic strain rate to the plastic yield surface is necessary and sufficient for the constitutive inequality. In viscoplasticity, however, given zero-order homogeneity and the yield conditions, normality is not a necessary condition. However, in conjunction with the restriction expressed by equation (32), it is a sufficient condition.

It is of interest to estimate the severity of the restriction equation (32) imposes on the range of choices for the yield surface function. After Tsai and Wu[16], we treat an anisotropic yield function

$$F = \sqrt{[A_{ijkl}\sigma_{ij}\sigma_{kl} + A_{ij}\sigma_{ij}]} - 1. \tag{33}$$

Here we regard the coefficients A_{ijkl} and A_{ij} as independent of the stress but they may in general depend on the dependent state variables $\varepsilon_{ij}{}^{i}$ and k.

We will consider the case in which

$$\phi(F) = F, \quad \text{so that}$$

$$\langle\phi(F)\rangle = \langle F\rangle \geq 0, \quad \text{and}$$

$$\frac{\partial \phi}{\partial F} = 1 \geq 0.$$

We seek to determine the restriction equation (32) imposes on the coefficients of equation (33).

Computation yields that

$$\frac{\partial^2 F}{\partial \sigma_{ij}\,\partial \sigma_{pq}} = -\frac{[2A_{pqrs}\sigma_{rs} + A_{pq}][2A_{ijkl}\sigma_{kl} + A_{ij}]}{4\sqrt{(A_{mncd}\sigma_{mn}\sigma_{cd} + A_{ab}\sigma_{ab})^3}} + \frac{A_{pqij}\cdot}{\sqrt{\{A_{mnrs}\sigma_{mn}\sigma_{rs} + A_{ab}\sigma_{ab}\}}}.$$

Upon comparison with equation (32), we find

$$\begin{aligned} {}_{(1)}L_A + {}_{(1)}L_B \\ &= \eta\left[1 - \left\{\sqrt{(A_{mnrs}\sigma_{mn}\sigma_{rs} + A_{ab}\,\sigma_{ab})} - 1\right\}\left\{\frac{1}{\sqrt{(A_{mnrs}\sigma_{mn}\sigma_{rs} + A_{cd}\sigma_{cd})}}\right\}\right] \\ &= \left\{\eta\middle/\sqrt{(A_{mnrs}\sigma_{mn}\sigma_{rs} + A_{ab}\sigma_{ab})}\right\} \geq 0. \end{aligned}$$

To satisfy the form of equation (32) it is now only necessary that

$$\begin{aligned} &\eta\left\{\frac{\sqrt{(A_{mnrs}\sigma_{mn}\sigma_{rs} + A_{cd}\sigma_{cd})} - 1}{\sqrt{A_{abfg}\sigma_{ab}\sigma_{fg} + A_{fg}\sigma_{fg}}}\right\}A_{ijpq} \\ &= {}_{(0)}L\left[\frac{\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}}{2}\right] + \sum_{r=2}^{N}\{{}_{(r)}L_{(r)}b_{ij\,(r)}b_{pq}\}. \end{aligned} \tag{34}$$

Equation (34) implies the form

$$A_{ijpq} = \Gamma\left[\frac{\delta_{ip}\delta_{iq} + \delta_{iq}\delta_{jp}}{2}\right] + \sum_{r=2}^{N}\{{}_{(r)}B_{ij\,(r)}B_{pq}\} \tag{35}$$

where Γ and ${}_{(r)}B_{ij}$ respectively are positive and symmetric tensor functions of the dependent state variables, $\varepsilon_{ij}{}^{i}$ and k.

It is important to note that equation (32) implies no restriction on A_{ij}, the linear term in equation (33). This term represents the Bauschinger effect.

To understand equation (35) we will give attention to a special case of anisotropy. Referring to Hill[17] and Hoffman[18], we write a yield condition for an inelastically incompressible, transversely isotropic material without Bauschinger effect, which is in a plane stress state so that $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, where the y-z plane is the isotropic plane.

$$F = \sqrt{\left[\frac{\sigma_{yy}{}^2}{\Psi_y{}^2} + \frac{\sigma_{xx}{}^2}{\Psi_x{}^2} - \frac{\sigma_{xx}\sigma_{yy}}{\Psi_x{}^2} + \frac{\sigma_{xy}{}^2}{S_{xy}{}^2}\right]} - 1 \tag{36}$$

where

Ψ_y is the tensile yield stress in the y direction

Ψ_x is the tensile yield stress in the x direction

and

S_{xy} is the in-plane pure shear yield stress.

Referring to both equation (36) and equation (35) we obtain

$$\begin{aligned} A_{xxxx} &= 1/\Psi_x^2 = \Gamma + B_{xx}^2 \\ A_{yyyy} &= 1/\Psi_y^2 = \Gamma + B_{yy}^2 \\ A_{xxyy} &= 1/\Psi_x^2 = B_{xx} B_{yy} \\ A_{xy} A_{xy} &= 1/S_{xy}^2 = B_{xy}^2. \end{aligned}$$

Eliminating in favor of B_{xx} yields

$$B_{xx}^4 - B_{xx}^2 \left[\frac{1}{\Psi_x^2} - \frac{1}{\Psi_y^2}\right] - \frac{1}{4\Psi_x^2} = 0,$$

for which one obtains the solution

$$B_{xx}^2 = \frac{\left\{\frac{1}{\Psi_x^2} - \frac{1}{\Psi_y^2}\right\} \pm \sqrt{\left[\left\{\frac{1}{\Psi_x^2} - \frac{1}{\Psi_y^2}\right\}^2 + \frac{1}{\Psi_x^2}\right]}}{2}.$$

The positive sign solution yields real values of B_{xx} irrespective of the relative magnitudes of Ψ_x and Ψ_y. It is elementary to recognize that this result applies also for the solution for B_{yy}, Γ and B_{xy}. Hence, the yield function in the present special case satisfies the constitutive inequality without restriction on the relative magnitude of the yield stresses.

The Hohenemser–Prager[3] prototype of viscoplastic constitutive relations is recovered if one sets:

(a) $\dot{\varepsilon}_{ij}^{e} = 0$

(b) $\dot{\varepsilon}_{ij}^{i} \delta_{ij} = 0$

(c) $F = \sqrt{\left(\frac{s_{kl} s_{kl}}{2k_0}\right)} - 1$

where s_{ij} is the deviatoric stress tensor and k_0 is a constant. Note that, given (b) above,

$$\frac{\partial^2 F}{\partial \sigma_{ij}\, \partial \sigma_{pq}} = \frac{\partial^2 F}{\partial s_{ij}\, \partial s_{pq}}$$

and that (c) in conjunction with equation (36) implies the relations

$$\begin{aligned} S_{xy} &= k_0 \\ \Psi_x = \Psi_y &= \sqrt{3}\, k_0. \end{aligned}$$

Summarizing, it does not appear that equation (32) implies very severe restrictions on the range of choices for the yield surface function.

REFERENCES

1. E. C. Bingham, *Fluidity and Plasticity*. McGraw-Hill, New York (1922).
2. H. Hencky, Ueber langsaeme stationaze Strömumg in Plastischen Massen, *Z.A.M.M.* **5**, 115 (1925).
3. K. Hohememser and W. Prager, Über die Ansätze der Mechanik Isotroper Kontinua, *Z.A.M.M.* **12**, 216–226 (1932).
4. P. Perzyna, Fundamental Problems in Viscoplasticity, *Advances in Applied Mechanics*, Vol. 9. Academic Press (1966).

5. A. Phillips, and H. C. Wu, A theory of viscoplasticity, *Int. J. Solids Struct.* **9**, 15–30 (1973).
6. J. Lubliner, A generalized theory of strain rate dependent plastic wave propagation in bars, *J. Mech. Phys. Sol.*, 59–66 (1964).
7. N. Cristescu, *Dynamic Plasticity*, North-Holland Publ. Co., Amsterdam (1968).
8. W. Prager, Recent developments in the theory of plasticity, *J. appl. Phys.* **20**, 235–241 (1949).
9. D. C. Drucker, A more fundamental approach to plastic stress strain relations, *Proc. 1st U.S. Nat. Congress Applied Mechanics*, pp. 487–491 (1951).
10. A. Phillips and M. A. Eisenberg, Observations on certain inequality conditions in plasticity, *Int. J. Non-Linear Mechanics*, **1**, 247–256 (1967).
11. P. M. Naghdi, *Stress–Strain Relations in Plasticity and Thermoplasticity*, *Plasticity*, Pergamon Press, pp. 121–169 (1960).
12. D. C. Drucker, A definition of stable inelastic material, *J. appl. Mech.*, 101–106 (1959).
13. J. F. Bell, *The Physics of Large Deformation of Crystalline Solids*, Springer-Verlag (1968), pp. 241–254 (1967).
14. E. R. Wood, and A. Phillips, On the theory of plastic wave propagation in a bar, *J. Mech. Phys. Sol.* 241–254, (1967).
15. J. B. Martin, A note on the uniqueness of solutions for dynamically loaded rigid–plastic and rigid–viscoplastic continua, *J. appl. Mech.*, 207–209 (1966).
16. S. Tsai, and E. Wu, A general theory of strength of anisotropic materials, *J. Comp. Mat.* **5**, 58–81 (1971).
17. R. Hill, *The Mathematical Theory of Plasticity*. The Clarendon Press, Oxford, p. 317 (1950).
18. O. Hoffman, The brittle strength of orthotropic materials, *J. Comp. Mat.* **1**, 201–207 (1967).

Абстракт—В разных формулировках теории пластичности очевидной является структура, принимающая некоторые характерные признаки, заключающие идеальную текучесть, условие текучести и неравенство состояния. Посредством этих признаков определяются уравнения пластичности. В предлагаемой работе вводится дополняющая вязкопластическая часть неравенства состояния теории пластичности и рассматривается ее физические значения. Дается, также, теория вязкопластичности, обладающая подобной структурой к такой же в пластичности. Исследуется связь между теорией вязкопластичности и прототипом вязкопластических соотношений состояния Гогенемзера — Прагера.